REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
15-06-2008	Final Performance Report	01-11-2004-31-03-2008

4. TITLE AND SUBTITLE
Multi-scale Analysis of Heterogeneous Media

5a. CONTRACT NUMBER

5b. GRANT NUMBER
FA9550-05-1-0008

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
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5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
USAF, AFRL
AF Office of Scientific Research
875 N. Randolph St. Room 3112
Arlington, VA 22203

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Distribution Statement A: Unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT
The project develops new mathematical tools for the description of multi-scale stress transfer inside composite materials. The first research activity provides new mathematical methods to characterize the extreme local stress excursions inside linear elastic composite materials. Research in this area has lead to new asymptotic formulations for the local stress excursions across lamina interfaces inside fiber reinforced laminates. New fast computational methods for calculating bounds on the extreme local stress excursions inside prestressed composites are developed using the asymptotic expansions. The second research track examines the effects of elasto-plasticity on the extreme local stress excursions inside composite media. New upper bounds on the strength domain of the composite are found that incorporate multi-scale information on the norms of solutions to nonlinear elasticity and conductivity problems. The third research activity develops a novel and systematic computational design method for the design of composite structures that hedge against structural failure. The fourth research thrust characterizes the influence of the boundary data on the local stress fluctuations inside microstructured media. New rigorous lower bounds on the boundary layer decay of stress fluctuations with respect to the length scale of oscillation in the boundary data have been obtained.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
U	U	U	UU		Robert Lipton
					19b. TELEPHONE NUMBER *(Include area code)* 225 578 1569

Contents

1 Overview

The goal of this project is to develop new mathematical tools for the quantitative description of multi-scale stress transfer inside composite materials. The first research thrust seeks to refine the linear elastic stress analysis developed under previous AFOSR support in [18], [19] and [20] by incorporating strain gradient effects and higher order boundary layer effects in the vicinity of high stress gradients. Such gradients occur in the neighborhood of joints, bolts and rivets as well as across interfaces where there is an abrupt jump in microstructure. Research work related to this objective has been carried out by the PI in collaboration with Dr. Ivo Babuska (University of Texas) and Dr. Michael Stuebner (LSU and NCSU). This research focuses on characterizing the influence of the boundary data on the local stress fluctuations inside microstructured media. New estimates for the boundary layer decay of stress fluctuations with respect to the length scale of oscillation in the boundary data have been obtained.

The second research activity seeks to characterize extreme local stress excursions inside linear elastic composite materials in the presence of prestress and in the vicinity of rapid microstructure variation. Research in this area has lead to new asymptotic formulations for the local stress excursions across lamina interfaces inside fiber reinforced laminates. This work is carried out in collaboration with Dr. Endel Iarve (University if Dayton) and Dr. Tim Breitzman (Air Force Research Laboratory). New fast computational methods for calculating bounds on the extreme local stress excursions inside prestressed composites are developed using the asymptotic expansions. These methods have been coded into subroutines and are now being used on a trial basis at Sikorski Aircraft Company. The results obtained from this activity formed the basis for the Ph.D. Thesis of Timothy Breitzman.

The third research track examines the effects of elasto-plasticity on the extreme local stress excursions inside composite media. The PI has been working with his Ph.D. students Silvia Jimenez and Bacim Alali on different aspects of this project over years two and three. New lower bounds are obtained on the Sobolev norms of solutions to nonlinear elasticity and conductivity problems. These provide upper bounds on the strength domain of the composite.

The fourth research activity applies the aforementioned efforts to develop computational design strategies for composite structures that hedge against structural failure. Here the opportunity for failure is minimized through the suitable tailoring of the composite microstructure. The objective is to design a gradation in the microstructure in order to minimize the extent of high stress zones generated by holes or sharp changes in boundary loading. This project has been successful and numerical algorithms have been developed for the design of graded microstructure in the presence of bolt holes and reentrant corners. This work is carried out together with Dr. Michael Stuebner (LSU and NCSU) over years one through three. The results obtained from this research activity formed the basis for the Ph.D. Thesis of Michael Stuebner.

Additionally during 2006-2007 the PI was able to extend the scope of research to successfully develop mathematical methods for the analysis of nonlocal models of elastic interaction in composite media. This extension was motivated by the PI's research results obtained under previous AFOSR funding together with the results obtained in the second research thrust area of this project. After meeting with scientists at Boeing Phantom Works it was determined that the PI's AFOSR supported results were able to rigorously justify and

extend well known formal procedures used at Boeing for modeling stress excursions in composites. Based on this the PI was invited to develop multi-scale analysis techniques for the nonlocal theory of elasticity known as Peridynamics. This research area is also being supported through a contract with Boeing Company. The support period started the Fall of 2006 and runs through Spring 2008. This work is carried out together with the PI's Ph.D. student Bacim Alali. The PI benefits greatly from the scientific interaction between the PI and Dr. Abe Askari (Boeing) and Dr. Stewart Silling (Sandia) on this project.

Three Ph.D. theses have followed from the research supported under this grant: Dr. Timothy Breitzman is now a research scientist at the Air Force Research Laboratory Materials Directorate at Wright Patterson AFB in Ohio and Dr. Michael Stuebner is now a Postdoc with the Mathematics of Materials Research Training Group at North Carolina State University. Mr. Bacim Alali will be receiving his Ph.D. August 2008 and will be joining the Faculty in the Department of Mathematics at the University of Utah as a Wiley Assistant Professor.

2 Summary of Research Projects

2.1 Optimal lower bounds on the local stress and strain inside random two-phase composites subjected to macroscopic loading

The majority of work on composite media has focused on the characterization of effective overall properties of the medium. Such properties include the effective elastic, electric, magnetic and thermal properties of a composite. The effective properties of the medium are bounded or estimated in terms quantities that depend on the partial statistical description of the microstructure, see for example the monograph [22]. Bounds on effective properties predict the worst or best overall response that can be expected from an ensemble of configurations described by the available statistical information.

It is very important to complement the theory of effective properties with new techniques for teasing out the relationships that connect the local field behavior inside a body containing random heterogeneous media and the macroscopic or long wavelength loads that are applied to the body. These relationships provide a way to assess the likelihood of failure due to the presence of extreme local fields generated by an applied macroscopic load. Quantities that are sensitive to extreme local gradients and fluxes inside composite media are given by L^p norms of the local fields for $p \geq 2$. These objects can detect extreme field behavior inside composite media that can not detected from bounds or estimates of effective properties. Recently the PI's AFOSR supported research has lead to the development of the first optimal lower bounds that apply to all L^p norms of local thermal gradients and electric fields for $p \geq 2$. The bounds apply to two-phase composites and are given in terms of two-point correlation functions and the volume fractions of the two-phase medium, see [14]. These bounds provide an explicit link between applied macroscopic field gradients, the two point correlation function describing the random geometry, and the local field gradients inside random composites.

Under this support period the PI's research has lead to the first optimal lower bounds on the L^p, $2 \leq p \leq \infty$, norms of the rotational invariants of local stress and strain tensors inside random elastic composite materials. The bounds provide a new opportunity for the assessment of load transfer between macroscopic and microscopic length scales for

statistically defined microstructures. The new lower bounds on the maximum point wise hydrostatic stress and deviatoric stress provide explicit conditions on the macroscopic applied stress for which the local stress lies out side the strength domain of each material phase. These results are anticipated to have wide impact in the field of strength prediction for random composite media.

2.1.1 Optimal lower bounds on the local hydrostatic stress and strain inside random two-phase composites subjected to hydrostatic loading

Failure initiation in composite materials is a multi-scale phenomena. Central to the analysis is the assessment of the local stress and strain fields generated by macroscopic forces. Quantities sensitive to local field behavior include higher order moments of the stress and strain fields inside the composite. These quantities have seen extensive application in the theoretical analysis of material failure, see Kelly and Macmillan [11]. Failure criteria for fiber reinforced composites are often associated with the deviatoric part of the elastic strain tensor. However critical dilatational deformation can precede critical deviatoric deformation in polymers, see Asp, Berglund and Talerja [2]. The dilatational strain has recently been incorporated into failure criteria for epoxy matrix composites seen in aircraft, see Gosse and Christensen [8].

Composites made from two linear isotropic elastic materials are considered. It is assumed that only the volume fraction of each elastic material is known. The composite is subjected to a uniform hydrostatic strain. For this case lower bounds on all r^{th} moments of the dilatational strain field inside each phase are obtained for $r \geq 2$. A lower bound on the maximum value of the dilatational strain field is also obtained. These bounds are given in terms of the volume fractions of the component materials. All of these bounds are shown to be the best possible as they are attained by the dilatational strain field inside the Hashin–Shtrikman coated sphere assemblage. The bounds provide a new opportunity for the assessment of the local dilatational strain in terms of a statistical description of the microstructure. These results are presented in R. Lipton, "Optimal lower bounds on the dilatational strain inside random two-phase elastic composites subjected to hydrostatic loading," *Mechanics of Materials*, **38**, 2006, pp. 833-839.

When the composite is subjected to a uniform hydrostatic stress we examine higher order moments of the local hydrostatic stress to gage the effect of the microstructure on the amplification of the macroscopic stress inside random composites made from two isotropic elastic materials in prescribed proportions. Lower bounds on all r^{th} moments of the hydrostatic stress field inside each phase are obtained for $r \geq 2$. These are given in terms of the volume fractions of the constituent materials. A lower bound on the maximum value of the hydrostatic stress field is obtained when only the volume fractions of the constituents are known. All of these bounds are shown to be the best possible as they are attained by the hydrostatic stress field inside the Hashin–Shtrikman coated sphere assemblage.

The bounds provide a new opportunity for the assessment of load transfer between macroscopic and microscopic scales for statistically defined microstructures. These results are presented in R. Lipton, "Optimal lower bounds on the hydrostatic stress amplification inside random two-phase elastic composites." *Journal of the Mechanics and Physics of Solids*, **53**, 2005, pp. 2471–2481.

2.1.2 Optimal lower bounds on the local stress and strain inside random two-phase composites subjected to shear loading and mixed mode loading

In this project we extend the theory to situations involving general macroscopic loading conditions. Composites made from two linear isotropic elastic materials are considered. It is assumed that only the volume fraction and elastic properties of each elastic material are known. We are able to find optimal lower bounds on the local stress and strain for several different loading conditions. The cases covered by this analysis do not yet provide the full story but they are significant and necessary for further developments in this area. The cases covered by this analysis include:

- Optimal lower bounds on the norms of the local stress and strain tensors when the sample is subjected to hydrostatic macroscopic stress or strain loading.
- Optimal lower bounds on the norms of the local stress and strain for a subspace of mixed mode loading characterized by special a dimensionless group of material parameters.
- Optimal lower bounds for the local shear stress and shear strain when the macroscopic loading is a pure shear stress or strain and the bulk moduli of the two materials are the same.
- Optimal lower bounds on the hydrostatic component of the local stress and strain for general applied macroscopic loading when the bulk moduli of the two materials are the same.
- Optimal lower bounds on the hydrostatic component of the local stress and strain for general applied macroscopic loading and the shear moduli of the two components are the same.

The microgeometries that attain the bounds depend upon the macroscopic loading and material properties. In this project we identify several distinguished parameter regimes where the optimal configurations are given by layered materials, coated sphere constructions, or coated confocal ellipsoid constructions.

This work is joint with the PI's Ph.D. student Bacim Alali who is being supported by this grant. These results are in preprint form and will be submitted for publication in 2008.

2.2 Optimal design subject to point wise stress constraints through inverse homogenization

Modern design practice increasingly incorporates the use of load bearing components made from composite materials. Composites are now used in structural geometries that involve abrupt dimensional changes within structural components, such as skins connected to ribs, panel reinforcements and junctions of struts. Associated with these geometries are stress concentrations and the potential for failure. In this project new higher order homogenization results are developed and applied in an inverse homogenization procedure to numerically design graded microstructures that provide desirable structural response while confining the effects of stress concentrations generated along joints or junctions between structural elements.

The computational design method is based the macrostress modulation functions introduced by the PI [18], [19], [20] under previous AFOSR support. These quantities are used by the PI to develop a rigorous theory for the design of continuously graded locally periodic microstructures in [21]. The design method for continuously graded locally layered microstructures is worked out in collaboration with the PI's recent Ph.D. student Dr. Michael Stuebner. In this project the explicit formulas for the modulation functions are derived for layered materials and coated cylinder assemblages.

The methodology is first illustrated for long cylindrical shafts reinforced with stiff cylindrical elastic fibers with generators parallel to the shaft. The local fiber geometry can change across the shaft cross section. The methodology is implemented numerically for a cross–sectional shape that possesses reentrant corners typically seen in lap joints and junctions of struts. Graded locally periodic fiber reinforced microgeometies are recovered that provide the necessary structural rigidity with respect to torsion loading while at the same time minimizing the spatial extent of high stress zones generated by stress concentrations at the reentrant corners. In the second and third years the numerical algorithm has been extended to handle the graded material design problem for two-dimensional elastic problems of plane strain and plane stress. Computational examples have been carried out illustrating the utility of the numerical method for the graded material design of flanges. This work is in collaboration with Dr. Michael Stuebner now at North Carolina State University and formed the basis of his Ph.D. Thesis at LSU. The results of this research effort have been reported in the following publications:

- Lipton R. and Stuebner M. "Inverse homogenization and design of microstructure for point wise stress control." *Quarterly Journal of Mechanics and Applied Mathematics*, **59**, 2006, pp. 139–161.
- Lipton R. and Stuebner M. "Optimization of composite structures subject to local stress constraints." *Computer Methods in Applied Mechanics and Engineering*, **196**, 2006, pp. 66–75.
- Lipton R. and Stuebner M. "Optimal design of composite structures for strength and stiffness: an inverse homogenization approach." *Structural and Multidisciplinary Optimization*, **33** 2007, pp. 351–362.

2.3 Stress and strain analysis for prestressed composites

It is well known that homogenization theory provides the methodology for understanding the behavior of averaged fields associated with fine scale structure. However new techniques are required in order to rigorously capture the local excursions of the stress about the homogenized value. In years two and three we studied the effect of rapid change in microstructure on the local field behavior inside composite materials with residual stress. Such rapid variation of microstructure is typical in engineering composites including fiber reinforced laminated materials and braided fiber reinforced composites. This investigator together with Dr. Timothy Breitzman and Dr. Endel Iarve have developed the necessary rigorous asymptotic analysis that characterizes the local behavior of stress and strain fields at the interface between plys in fiber reinforced laminates. The mathematical techniques are general and can be used for any prestressed composite system exhibiting abrupt changes in microgeometry.

- The analysis is based on a physically motivated anzatz developed by the PI for characterizing the local field behavior at the ply interface inside composites with residual stress.
- This anzatz is shown to give the correct point wise asymptotics of the stress and strain fields in the regime when the fiber spacing and diameter are significantly smaller than the ply thickness.
- The method employs a new application of compensated compactness for suitably constructed div-curl products.

To the best of the PI's knowledge this is the first such method in the mathematical or engineering literature for the description of local field behavior in zones where the microstructure changes abruptly. This work has been used to develop a fast algorithm for multi-scale stress analysis in fiber reinforced laminates across ply interfaces. This work is carried out in collaboration with Dr. Timothy Breitzman and Dr. Endel Iarve at the Air Force Research Laboratory at Wright Patterson Air Force Base. Numerical studies show that the fast algorithm faithfully captures the trends seen in the much more expensive direct numerical simulation.

In order to place these research results within a greater context we note that the presence of residual stress inside composite materials is an important factor that influences the processing and design of structural components [5], [6], [9], and [10]. Its effects are spread across several length scales, the smallest being the fiber-matrix length scale, the next being the inter laminar length scale and the largest being the structural length scale. The net result of these effects can be seen in the warpage of autoclaved composite parts [1]. The recent work of [27] models each ply as a homogeneous orthotropic material and examines the effect of fiber prestress at inter laminar length scales while the numerical and experimental work presented in [3] details the effects of residual stress at the fiber-matrix length scale. The numerical analysis presented in this research work is complementary to these efforts. Here we have considered the combined effects of residual stress at the fiber-matrix length scale and at the inter laminar length scale and have presented a numerical procedure to resolve the local strain field at the fiber-matrix length scale inside each ply.

These results are published in SIAM Multiscale Modeling and Simulation **6** 2007, pp. 937–962.

2.4 Field concentrations and homogenization

A multi-scale characterization of the field concentrations inside composite and polycrystalline media is developed. We focus on gradient fields associated with the intensive quantities given by the temperature and the electric potential. In the linear regime these quantities are modeled by the solution of a second order elliptic partial differential equation with oscillatory coefficients. The characteristic length scale of the heterogeneity relative to the sample size is denoted by ε and the intensive quantity is denoted by u^ε. Field concentrations are measured using the L^p norm of the gradient field $\|\nabla u^\varepsilon\|_{L^p}$ for $2 \leq p < \infty$. The analysis focuses on the case when $0 < \varepsilon \ll 1$. Explicit lower bounds on $\liminf_{\varepsilon \to 0} \|\nabla u^\varepsilon\|_{L^p}$ are developed. The lower bounds are described in terms of the p^{th} order moments of the solution of two-scale corrector problems. The quantities are sensitive to microscopic field concentrations and can become divergent for $p > 2$. These bounds provide a way to rigorously

assess field concentrations generated by the microgeometry without having to compute the actual field u^{ε}. The analysis is carried out with minimal regularity assumptions on the coefficients describing the local properties inside the heterogeneous media. This work delivers the rigorous mathematical connection between the field concentration exponents introduced in Milton [23] and weakly convergent sequences of gradient fields associated with homogenization. This project provides illustrative examples of microstructures with homogenized gradients bounded in L^p for all $p \geq 2$ but for which

$$\liminf_{\varepsilon \to 0} \|\nabla u^{\varepsilon}\|_{L^p(D)} = \infty$$

for every $p > p_c$, where p_c is the field concentration exponent for the microstructure. The results of this project have appeared in, Lipton R. "Homogenization and field concentrations in heterogeneous media." *SIAM J. on Math. Analysis*, **38**, 2006, pp. 1048–1059.

2.5 Heterogeneous mixtures of nonlinear materials and field concentrations

In this project we build on the previous results for linear materials and develop tools for the assessmant of field concentrations inside mixtures of nonlinear media. Here the goal is to characterize field concentrations for weakly converging sequences of gradient fields associated with nonlinear elliptic differential equations with oscillatory coefficients and exponents. In this way we seek to understand extreme field excursions in the physical context of elasto-plasticity.

A canonical example is given by the family of equations

$$-\mathrm{div}\, a(\mathbf{x}/\epsilon, \nabla u^{\epsilon}) = f$$

where $a(\mathbf{y}, \zeta)$ is a periodic function of $\mathbf{y}$ in R^3 for every ζ in R^3. Here $a(\mathbf{y}, \zeta) = \alpha_1 |\zeta|^{p_1} \zeta$ for $\mathbf{y}$ in material one and $a(\mathbf{y}, \zeta) = \alpha_2 |\zeta|^{p_2} \zeta$ for $\mathbf{y}$ in material two. The parameters α_1 and α_2 are positive and $p_1 > p_2 \geq 2$. This example is characteristic of homogenization problems for power law materials in the limit $\varepsilon \to 0$. In joint work with my Ph.D. student Silvia Jimenez we provide explicit lower bounds on $\liminf_{\varepsilon \to 0} \|\nabla u^{\varepsilon}\|_{L^p}$ for $p \geq 2$. The bounds are given in terms of the L^p norms of corrector problems posed over the unit period cell. We outline conditions on the microstructure, boundary data and right hand side f for which these lower bounds are tight. We are presently extending these results to stationary random microstructures and for mixtures of materials with more general power law behavior.

2.6 Effect of boundary data on field fluctuations

In this project we seek to understand how field fluctuations inside heterogeneous media depend on the boundary data. The goal is to characterize the extreme behavior of field fluctuations inside a heterogeneous medium subjected to a prescribed class of boundary data when the configuration of materials is only known statistically. For example a two-phase conducting medium fills a square domain Q of unit side length. The configuration of the two isotropic conductors can be arbitrary but the proportions of each are fixed. This class of configurations is denoted by $\mathcal{C}$. Let $\mathcal{N}$ denote a prescribed class of Neumann data given on the boundary of Q. Consider next a concentric square Q_d of side length $d < 1$ inside Q. Let u be the temperature inside the two-phase conductor solving Fourier's law

of heat conduction. Here u satisfies Neumann boundary conditions with data g taken from the prescribed class $\mathcal{N}$. In this project we are interested in the penetration function p given by

$$p = \sup_{g \text{ in } \mathcal{N}} \sup_{\mathcal{C}} \frac{\int_{Q_d} |\nabla u|^2 \, dx}{\|g\|_{H^{-1/2}}}.$$

This function measures the extreme response of the field fluctuation $\int_{Q_d} |\nabla u|^2 \, dx$ relative to the $H^{-1/2}$ norm of the boundary data over the prescribed classes of boundary data and configurations. This quantity naturally appears in several contexts and can be thought of as a quantity that generalizes the notion of St Venant's principle. Understanding the behavior of p for different choices of $\mathcal{N}$ and $\mathcal{C}$ has been a joint collaboration with Dr. Ivo Babuska in years two and three. In recent work the class of Neuman data is taken to be subspaces of functions spanned by linear combinations harmonic polynomials, ie., the real and imaginary parts of z^n for $z = x + iy$. Here the subspace is chosen to be the span of all harmonic polynomials of degree greater than N. This choice of Neuman data is motivated by problems where the first N linear combinations of harmonic polynomials comprising the boundary data are known and the unknown part is confined to the span of the higher order harmonics. This situation arises when one wishes to calculate error estimates for generalized finite element methods based on partitions of unity see [4]. It also arises in the context of random media when the boundary data is also random and given in terms of truncated Karhunen-Loeve expansions [7]. For this case we are able to calculate lower bounds on the relative response of field fluctuations to determine that p decays no faster than $N^{-5/2}$. This is in dramatic contrast to a homogeneous isotropic conducting material where p decays exponentially with N. This is joint work with Dr. Ivo Babuska and Dr. Michael Stuebner and a manuscript is in preparation for publication in 2008.

2.7 Multi-scale peridynamic theory for heterogeneous media

During 2006-2007 the PI was able to extend the scope of research to successfully develop mathematical methods for the analysis of nonlocal models of elastic interaction in composite media. The nonlocal theory of elasticity treated in this project is known as as the peridynamic model [25] and [26]. The peridynamic theory may be thought of as a continuum version of molecular dynamics. The acceleration of any particle at $\mathbf{x}$ in the reference configuration at time t is found from the equation of motion,

$$\rho \partial_{tt}^2 \mathbf{u}(\mathbf{x}, \mathbf{t}) \quad = \quad L[\mathbf{u}(\mathbf{x}, \mathbf{t}), \mathbf{x}] + \mathbf{b}(\mathbf{x}, \mathbf{t}) \tag{1}$$

in which

$$L[\mathbf{u}(\mathbf{x}, \mathbf{t}), \mathbf{x}] \quad = \quad \int_{H_\delta} \mathbf{f}(\mathbf{u}(\mathbf{x}', \mathbf{t}) - \mathbf{u}(\mathbf{x}, \mathbf{t}), \mathbf{x}' - \mathbf{x}) \, \mathbf{dV}_{\mathbf{x}'} \tag{2}$$

where H_δ is a neighborhood of $\mathbf{x}$ of diameter δ, $\mathbf{u}$ is the displacement vector field, the function $\mathbf{f}$ gives the value of the force per unit volume squared that a particle at $\mathbf{x}'$ exerts on a particle at $\mathbf{x}$ and $\mathbf{b}$ is the body force. The function $\mathbf{f}$ can be thought of as providing the constitutive law for the material. The parameter δ is the horizon inside which particles interact and is a length scale associated with the peridynamic theory. In heterogeneous media made from two different peridynamic materials the function $\mathbf{f}$ depends on the location

of the particles at $\mathbf{x}'$ and $\mathbf{x}$ in the reference configuration. In this project we consider periodic fiber or particle reinforced heterogeneous materials. The phase separating the particles or fibers is commonly referred to as the matrix phase. The scale of the period is denoted by ε and we are interested in the asymptotic behavior of the solutions of the peridynamic equations denoted by $\mathbf{u}^{\varepsilon}$ in the limit as ε tends to zero. In what follows we suppose that the peridynamic horizon is on the length scale of the microstructure. The horizon is given by $\varepsilon\delta$ where δ is fixed. The peridynamic neighborhood $H_{\epsilon\delta}$ is given by a ball of diameter $\epsilon\delta$ and is of dimensions such that at most one matrix-inclusion interface can appear inside the peridynamic neighborhood of $\mathbf{x}$. We set $\zeta = \mathbf{x}' - \mathbf{x}$, $\eta = \mathbf{u}(\mathbf{x}', \mathbf{t}) - \mathbf{u}(\mathbf{x}, \mathbf{t})$ and in the fiber or particle phase we suppose that the pairwise force is linear in the bond stretch $s = \frac{|\eta+\zeta|-|\zeta|}{|\zeta|}$ and $\mathbf{f} = c_p s\mathbf{n}$ in the particle $\mathbf{f} = c_m s\mathbf{n}$, in the matrix, and $\mathbf{f} = c_i s\mathbf{n}$ across the interface. Here $\mathbf{n} = \frac{\eta+\zeta}{|\eta+\zeta|}$ and the constants c_p, c_m, c_i are the bond stiffnesses for the inclusions, matrix and interfaces respectively. Here c_p and c_m are given by the bulk modulus and Young's modulus of each of the phases where the Poisson ratio is fixed to be $1/4$. The constant c_i needs to be determined experimentally for the interface. In what follows we linearize $\mathbf{f}$ with respect to ζ and consider the linearized versions of $\mathbf{f}$ for the inclusions, matrix and interface. We apply the notions of weak convergence and two scale convergence to derive the homogenized peridynamic equations in the limit $\varepsilon = 0$. The homogenized equations are also peridynamic equations and are used to resolve the effects of oscillatory residual stresses due to matrix shrinkage under curing. We develop asymptotic expansions for $\mathbf{u}^{\varepsilon}$ using rescaled solutions of a suitable peridyamic equation posed on the unit period cell. The difference between the asymptotic expansions and the solution $\mathbf{u}^{\epsilon}$ converges in the L^p norm over bounded time intervals as $\varepsilon \to 0$. We provide explicit formulas for the convergence rate that are given in terms of the exponent of Hölder continuity for the non oscillatory parts of the initial data and body force. This work is carried out together with the PI's Ph.D. student Bacim Alali and is being prepared for publication. The PI benefits greatly from the scientific interaction with Dr. Abe Askari (Boeing) and Dr. Stewart Silling (Sandia) on this project.

3 Publications resulting from the supported research

1. Breitzman T., Lipton R, and Iarve E. "Local field assessment inside multiscale composite architectures." *SIAM Multiscale Modeling and Simulation,* **6** 2007, pp. 937–962.

2. Lipton R. and Stuebner M. "Optimal design of composite structures for strength and stiffness: an inverse homogenization approach." *Structural and Multidisciplinary Optimization*, **33** 2007, pp. 351–362.

3. Lipton R. "Homogenization and field concentrations in heterogeneous media." *SIAM J. on Math. Analysis*, **38**, 2006, pp. 1048–1059.

4. Lipton R. and Stuebner M. "Optimization of composite structures subject to local stress constraints." *Computer Methods in Applied Mechanics and Engineering*, **196**, 2006, pp. 66–75.

5. Lipton R. "Optimal lower bounds on the hydrostatic stress amplification inside random two-phase elastic composites." *Journal of the Mechanics and Physics of Solids*, **53**, 2005, pp. 2471–2481.

6. Lipton R. "Optimal lower bounds on the dilatational strain inside random two-phase elastic composites subjected to hydrostatic loading." *Mechanics of Materials*, **38**, 2006, pp. 833-839.

7. Lipton R. and Stuebner M. "Inverse homogenization and design of microstructure for pointwise stress control." *Quarterly Journal of Mechanics and Applied Mathematics*, **59**, 2006, pp. 139–161.

8. Zheng Z., Forest M. G., Lipton R., Zhou R., and Wang A. "Exact scaling laws for electrical conductivity properties of nematic polymer nano-composite monodomains." *Advanced Functional Materials*, **15**, 2005,

4 Interactions/Transitions

4.1 Interactions with Government Laboratories

Since May of 2000 the PI has been interacting with the research team in the Laboratory at Wright Patterson Air Force Base lead by Dr. Tia Benson Tolle. As of Spring of 2001 the PI has also been collaborating with Dr. Greg Schoeppner (WPAFB) and Dr. Endel Iarve (UDRI contractor with WPAFB. One of the principle objectives of this interaction is the characterization of stress concentrations in composite materials. More recently the PI together with Dr. Tim Breitzman (WPAFB), Dr. Endle Iarve and Dr. Greg Schoeppner have embarked on the development of a fast multiscale numerical method for accurate stress assessment in prestressed composite materials used in aircraft. This methodology is being applied to the design of composite repair patches for high performance aircraft. The stress analysis method developed under this support, together with E. Iarve (UDRI) and T. Breitzman (AFRL, WPAFB), has been incorporated into a micromechanical failure criterion evaluation algorithm and transitioned to Sikorski Aircraft, Stratford, CT. Point of contact at Sikorski Aircraft is Jeffrey R. Schaff Structural Methods and Prognostics Group Sikorski Aircraft. 6900 Main Street, Stratford, CT, Phone: 203-386-7423.

4.2 Pleanary Talks and Special Presentations

- "Composite Properties and Microstructure Part I: Effective Properties; Part II: Strength," at the IMA Tutorial/Workshop: Composites: Where Mathematics Meets Industry, February, 2005, Institute for Mathematics and its Applications University of Minnesota, Minneapolis, MN.
  http://www.ima.umn.edu/matter/winter/t2.html

- Workshop on Modeling, Analysis and Simulation of Multiscale Nonlinear Systems, Oregon State University, June 19-29, 2007.

- Boeing Phantom Works, Seattle, WA, May 2006. Presentation entitled, "Stress Transfer between macroscopic and microscopic length scales in random media."

4.3 Fellowships, Sabbatical and Short Term Invited Visits

- J. T. Oden Research Faculty Fellowship at the ICES at the University of Texas, June 2006, April 2007.

- IMA Tutorial/Workshop: Composites: Where Mathematics Meets Industry, February, 2005.
- Visiting Scholar, Division of Engineering and Applied Science, Harvard University, September 2004 – June 2005.

4.4 Supported Graduate Students and Post Doctoral Associates

- Postdoctoral adviser for Michael Stuebner, LSU, supported during the period 8/01/2006-5/31/2007. Dr. Stuebner now holds a 3 year Post Doctoral research position at North Carolina State University Department of Mathematics.
- Ph.D. adviser for Timothy Breitzman, LSU, supported during the period 9/01/2004-8/15/2005. Dr. Breitzman is now a Research Engineer at the Materials Directorate of the AFOSR Research Laboratory at Wright Patterson AFB.
- Ph.D. adviser for Bacim Alali, LSU, supported during the period 7/01/2007-11/31/2007. Mr. Alali will receive his Ph.D. August of 2008. Starting September of 2008 he will take the 3 year Wiley Instructorship position in the Mathematics Department at the University of Utah.

4.5 Presentations

Invited talks at academic institutions.

- Mathematics Colloquium, Iowa State University, Ames, Iowa, April 2007.
- Mathematics Colloquium, University of Kentucky, Lexington, Kentucky, April 2007.
- Mathematics Colloquium, University of Akron, Akron, OH, March 2007.
- Mathematics Colloquium, Oregon State University, Corvallis, OR, February 2007.
- Mathematics Colloquium, Washington State University, Pullman, WA, February 2007.
- Applied Mathematics Seminar, Washington State University, Pullman, WA, February 2007.
- ICES Seminar, University of Texas Austin, TX, February 2007.
- Applied Mathematics Seminar, Department of Mathematics, Texas A & M University, February 2006.
- Mathematics Colloquium, Department of Mathematics, Florida State University, March 2006.
- Applied Mechanics Colloquium, Division of Engineering and Applied Sciences, Harvard University, April 2005.
- Mathematics Colloquium, Mathematics Department, University of Kentucky, March 2005.
- Applied Mathematics Seminar, Tulane University, February, 2005.

- Mechanics Seminar, Department of Mechanical Engineering, Massachusetts Institute of Technology, February 2005.
- Department of Mechanical and Aerospace Engineering, University of Florida, February 2005.
- Analysis Seminar, Department of Mathematics, University of Pennsylvania, February 2005.

Invited talks at Conferences and Workshops.

- Symposium on Microstructure and PDE, 6th International Congress on Industrial and Applied Mathematics, Zurich, July 16-20, 2007.
- Conference on Inverse Problems Homogenization and Related Topics, January 2007, University of Central Florida.
- Special SIAM Session on Mathematics and Materials Science, 113th Annual Meeting of the American Mathematical Society, Joint Mathematics Meeting, New Orleans, LA, January 2007.
- AFOSR Joint Program Review, Long Beach, CA, August 2007.
- Challenges in Computer Simulations, National University of Singapore, July, 23-25, 2007.
- DOE Multiscale Mathematics and High-Performance Computing Summer School, Oregon State University, July 1-3, 2007.
- ASME-ASCE-SES Mechanics of Materials Conference, Austin, TX, June 2007.
- AMS Fall Central Section Meeting, University of Cincinnati, Cincinnati, OH, Oct. 2006.
- Seventh World Congress on Computational Mechanics, Los Angeles, CA, July 2006, Symposium on Mathematical and Computational Aspects of Multiscale and Multiphysics, and Symposium on Computational Bridging of Length Scales.
- SIAM Annual Meeting, Minysymposium on Microstructure and PDE, Boston, MA, July 2006.
- PNNL Workshop Multiscale Modeling of Materials, Tacoma WA May 2006.
- IUTAM symposium, TopoptSYMP2005, Rungstedgarrd, Copenhagen, Denmark, October 2005.
- AFOSR Joint Program Review, Long Beach, California, August 2005.
- 8^{th} U.S. National Congress on Computational Mechanics, Austin Texas, July 2005.
- Represented the American Mathematical Society at the 11^{th} annual Exhibition of the Coalition for National Science Foundation Funding on Capitol Hill, Washington DC, June 21, 2005. Presentation entitled, "Mathematics for Advanced Composites Technology."
  http://www.ams.org/government/cnsfex05.html.

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